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Notes on Quasivarieties and Maltsev Products

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Abstract

These notes were prepared for my research group in 2014. These results are not new. Most appear in similar form in [4]. Since neither [4] nor its English translation, [5] is easily accessible, it seemed worthwhile to make these notes available to a wider audience. The notation and terminology follows that of my book, [1].

Disciplines

Algebra | Mathematics

Comments

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NOTES ON QUASIVARIETIES AND MALTSEV PRODUCTS

CLIFFORD BERGMAN

These notes were prepared for my research group in 2014. These results are not new. Most appear in similar form in [4]. Since neither [4] nor its English translation, [5] is easily accessible, it seemed worthwhile to make these notes available to a wider audience. The notation and terminology follows that of my book, [1].

Definition 1. A *quasivariety* is a class of algebras closed under subalgebra, product, and ultraproduct. Equivalently (see [1, thm 5.4]) a class is a quasivariety iff closed under subalgebra and reduced product.

It is easy to see that the intersection of a family of quasivarieties is again a quasivariety. Thus we can talk about the quasivariety generated by a class of algebras.

Proposition 2. *Let $\mathcal{K}$ be a class of algebras. The quasivariety generated by $\mathcal{K}$ is $\mathsf{SPP_u}(\mathcal{K}) = \mathsf{SP_r}(\mathcal{K})$.*

A proof can be found in [2, thm. V.2.23] or [1, thm. 5.4].

Corollary 3. *Let $\mathbf{A}$ be a finite algebra. The quasivariety generated by $\mathbf{A}$ is $\mathsf{SP}(\mathbf{A})$.*

Definition 4. A *quasiidentity* is a formula of the form

$$\big(p_1(\mathbf{x}) \approx q_1(\mathbf{x})\big) \curlywedge \big(p_2(\mathbf{x}) \approx q_2(\mathbf{x})\big) \curlywedge \cdots \curlywedge \big(p_k(\mathbf{x}) \approx q_k(\mathbf{x})\big) \to s(\mathbf{x}) \approx t(\mathbf{x})$$

When $k = 0$ we have the identity $s(\mathbf{x}) \approx t(\mathbf{x})$, so every identity is a quasiidentity.

Theorem 5. *A class of algebras is a quasivariety if and only if it is defined by a set of quasiidentities.*

For a proof of Theorem 5 see [2, V.2.25].

Congruence classes play a double role in the context of Maltsev products: as elements of a quotient algebra and as (potential) subalgebras. In order to help keep things straight, let us write $[a]_\theta$ for a congruence class being treated as a subset, and continue to write a/θ for the corresponding element of the quotient algebra.

An element, a, of an algebra, $\mathbf{A}$, is called *idempotent* if $\{a\}$ forms a subuniverse of $\mathbf{A}$. Put another way, for every basic operation, f, we have

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$$
\begin{tikzcd}
\mathbf{A} \arrow[rr, "h"] \arrow[dr, "g_1"'] & & \mathbf{A}/\alpha \arrow[dl, "g_2"] \\
& \mathbf{A}/\beta &
\end{tikzcd}
$$

Figure 1

$f(a, a, a, \dots, a) = a$. The algebra $\mathbf{A}$ is idempotent if every element is idempotent. A class, $\mathcal{K}$, of algebras is idempotent if every member algebra is idempotent.

Definition 6. Let $\mathcal{A}$ and $\mathcal{B}$ be quasivarieties. Then

$$
\begin{aligned}
\mathcal{A} \circ \mathcal{B} = \{\, \mathbf{R} : (\exists \theta \in \operatorname{Con}(\mathbf{R}))\ \mathbf{R}/\theta \in \mathcal{B} \text{ and} \\
(\forall r \in R)\ [r]_\theta \in \operatorname{Sub}(\mathbf{R}) \implies [r]_\theta \in \mathcal{A} \,\}.
\end{aligned}
$$

The class $\mathcal{A} \circ \mathcal{B}$ is called the *Maltsev product* of $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{C}$ is another quasivariety containing both $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \circ_{\mathcal{C}} \mathcal{B} = \big(\mathcal{A} \circ \mathcal{B}\big) \cap \mathcal{C}$. For the extent of this paper, by an *$\mathcal{A}$,$\mathcal{B}$-pivot* (or just a pivot if the context is clear) we mean a congruence θ satisfying the conditions of Definition 6.

Let $h \colon \mathbf{A} \to \mathbf{B}$ be a homomorphism with kernel α. Let $r \in A$. Then $[r]_\alpha$ is a subalgebra of $\mathbf{A}$ if and only if $h(r)$ is idempotent in $\mathbf{B}$. This is an immediate consequence of the fact that a direct image of a subalgebra is a subalgebra and the inverse image of a subalgebra is a subalgebra. Another way to say this is

$$
[r]_\alpha \in \operatorname{Sub}(\mathbf{A}) \iff f(r, r, \dots, r)\ \alpha\ r \text{ for every basic operation } f. \tag{1}
$$

Now suppose that $\alpha \le \beta \in \operatorname{Con}(\mathbf{A})$. Then

$$
\begin{aligned}
[r]_\alpha \in \operatorname{Sub}(\mathbf{A}) \implies (\forall f)\ f(r, \dots, r)\ \alpha\ r \implies \\
(\forall f)\ f(r, \dots, r)\ \beta\ r \implies [r]_\beta \in \operatorname{Sub}(\mathbf{A})
\end{aligned}
\tag{2}
$$

in which the quantifier on f ranges over all basic operations of $\mathbf{A}$.

Here is another observation.

Lemma 7. *Let $\mathbf{A}$ be an algebra, $\alpha < \beta$ congruences on $\mathbf{A}$ and $r \in A$.*

(1) $[r/\alpha]_{\beta/\alpha} = \big([r]_\beta\big)/\alpha$.
(2) $[r]_\beta \in \operatorname{Sub}(\mathbf{A}) \iff [r/\alpha]_{\beta/\alpha} \in \operatorname{Sub}(\mathbf{A}/\alpha)$.

Proof. For (1), $x/\alpha \in [r/\alpha]_{\beta/\alpha} \iff x/\alpha \equiv r/\alpha \pmod{\beta/\alpha} \iff x \equiv r \pmod{\beta} \iff x \in [r]_\beta \iff x/\alpha \in [r]_\beta/\alpha$.

The second claim follows from equivalence (1) since

$$
[r]_\beta \in \operatorname{Sub}(\mathbf{A}) \iff f(r, r, \dots, r)\ \beta\ r \iff f(r, r, \dots, r)\ (\beta/\alpha)\ r/\alpha.
$$

□

Let $\mathcal{B}$ be a quasivariety and $\mathbf{R}$ an algebra of the same similarity type as $\mathcal{B}$. Define

$$\Lambda^{\mathbf{R}}_{\mathcal{B}} = \{\, \theta \in \mathrm{Con}(\mathbf{R}) : \mathbf{R}/\theta \in \mathcal{B} \,\}$$
$$\lambda^{\mathbf{R}}_{\mathcal{B}} = \bigcap \Lambda^{\mathbf{R}}_{\mathcal{B}}.$$

The congruence $\lambda^{\mathbf{R}}_{\mathcal{B}}$ is called the *verbal congruence on* $\mathbf{R}$ *induced by* $\mathcal{B}$. We leave off the sub- and superscript when the context is clear. Notice that $1_R \in \Lambda$ since $\mathcal{B}$ contains a trivial algebra. Observe also that

$$\mathbf{R}/\lambda \leq \prod_{\theta\in\Lambda} \mathbf{R}/\theta \in \mathsf{SP}(\mathcal{B}) = \mathcal{B}.$$

Thus $\lambda \in \Lambda$. In fact the verbal congruence is the smallest congruence on $\mathbf{R}$ whose induced quotient falls into the quasivariety $\mathcal{B}$.

Now suppose that $\mathbf{R} \in \mathcal{A} \circ \mathcal{B}$. Let θ be any $\mathcal{A}$,$\mathcal{B}$-pivot congruence on $\mathbf{R}$. Since $\mathbf{R}/\theta \in \mathcal{B}$ we have $\lambda_{\mathcal{B}} \leq \theta$. Consequently, for every $r \in R$, $[r]_\lambda \subseteq [r]_\theta$. Suppose that $[r]_\lambda \in \mathrm{Sub}(\mathbf{R})$. By implication (2) $[r]_\theta \in \mathrm{Sub}(\mathbf{R})$ hence $[r]_\lambda \leq [r]_\theta \in \mathcal{A}$ which implies $[r]_\lambda \in \mathcal{A}$. Thus, in Definition 6, we can always take the $\mathcal{A}$,$\mathcal{B}$-pivot to be $\lambda_{\mathcal{B}}$.

Lemma 8. *Let $\mathcal{A}$ and $\mathcal{B}$ be any two quasivarieties. Then $\mathcal{A} \circ \mathcal{B}$ is closed under subalgebra.*

Proof. Let $\mathbf{R} \in \mathcal{A}\circ\mathcal{B}$ and let θ be an $\mathcal{A}$,$\mathcal{B}$-pivot on $\mathbf{R}$. Let $\mathbf{S}$ be a subalgebra of $\mathbf{R}$. We must show $\mathbf{S} \in \mathcal{A} \circ \mathcal{B}$. Define $\psi = \theta\restriction_S$. Then ψ is a congruence on $\mathbf{S}$ and $\mathbf{S}/\psi \leq \mathbf{R}/\theta$. since $\mathcal{B}$ is closed under subs, $\mathbf{S}/\psi \in \mathcal{B}$.

Now let $t \in S$ and assume $[t]_\psi \in \mathrm{Sub}(\mathbf{S})$. We claim that $[t]_\theta \in \mathrm{Sub}(\mathbf{R})$. By equivalence (1)

$$[t]_\psi \in \mathrm{Sub}(\mathbf{S}) \implies f(t,\dots,t)\ \psi\ t \implies f(t,\dots,t)\ \theta\ t \implies [t]_\theta \in \mathrm{Sub}(\mathbf{R}).$$

Finally, since $[t]_\theta \in \mathrm{Sub}(\mathbf{R})$, $[t]_\theta \in \mathcal{A}$. But $\mathcal{A}$ is closed under subs and $[t]_\psi \leq [t]_\theta$, so $[t]_\psi \in \mathcal{A}$ as desired. □

Lemma 9. *Let $\mathcal{A}$ and $\mathcal{B}$ be any two quasivarieties of finite similarity type. Then $\mathcal{A}\circ\mathcal{B}$ is closed under reduced products. If $\mathcal{B}$ is idempotent, the requirement of finite similarity type can be dropped.*

Proof. Let $\mathbf{R}_i \in \mathcal{A} \circ \mathcal{B}$, for $i \in I$, and let $\mathcal{F}$ be a filter on I. We must show $\prod_I \mathbf{R}_i/\eta_{\mathcal{F}} \in \mathcal{A} \circ \mathcal{B}$. By assumption, for each $i \in I$ we have a pivot congruence, θ_i on $\mathbf{R}_i$. Let us write $\mathbf{R} = \prod_I \mathbf{R}_i$.

For every $\mathbf{a}, \mathbf{b} \in R$ define $J(\mathbf{a}, \mathbf{b}) = \{\, i \in I : (a_i, b_i) \in \theta_i \,\}$. Note that $J(\mathbf{a}, \mathbf{b}) \supseteq [\![\mathbf{a} = \mathbf{b}]\!]$. Let $\psi = \big\{\, (\mathbf{a}, \mathbf{b}) \in R^2 : J(\mathbf{a}, \mathbf{b}) \in \mathcal{F} \,\big\}$. It is easy to check that $\psi \in \mathrm{Con}(\mathbf{R})$ and that $\eta_{\mathcal{F}} \leq \psi$. By the correspondence theorem we have $\mathbf{R}/\psi \cong (\mathbf{R}/\eta_{\mathcal{F}})/(\psi/\eta_{\mathcal{F}})$.

Let us write $\overline{\mathbf{R}}$ in place of $\mathbf{R}/\eta_{\mathcal{F}}$, $\bar\psi$ for $\psi/\eta_{\mathcal{F}}$ and $\bar{\mathbf{r}}$ in place of $\mathbf{r}/\eta_{\mathcal{F}}$. Then the isomorphism in the previous paragraph can be rewritten as $\mathbf{R}/\psi \cong \overline{\mathbf{R}}/\bar\psi$. Our task is to show that $\overline{\mathbf{R}} \in \mathcal{A} \circ \mathcal{B}$. $\bar\psi$ will be the pivot congruence on $\overline{\mathbf{R}}$ that makes this happen.

Let h be the composite of the natural maps $\mathbf{R} \twoheadrightarrow \prod(\mathbf{R}_i/\theta_i) \twoheadrightarrow \prod(\mathbf{R}_i/\theta_i)/\eta_{\mathcal{F}}$. Then h is surjective and unwinding the definition shows that $\ker(h) = \psi$. Thus

$$\overline{\mathbf{R}}/\bar{\psi} \cong \mathbf{R}/\psi \cong \prod(\mathbf{R}_i/\theta_i)/\eta_{\mathcal{F}} \in \mathcal{B} \tag{3}$$

since $\mathcal{B}$ is closed under reduced products.

Now let $\bar{\mathbf{r}} \in \overline{\mathbf{R}}$ and suppose that $[\bar{\mathbf{r}}]_{\bar{\psi}}$ is a subuniverse of $\overline{\mathbf{R}}$. We must show that $[\bar{\mathbf{r}}]_{\bar{\psi}} \in \mathcal{A}$. Let $\mathbf{r}$ be an element of R such that $\mathbf{r}/\eta_{\mathcal{F}} = \bar{\mathbf{r}}$. Note that $\mathbf{r}$ is not unique. By Lemma 7, $[\bar{\mathbf{r}}]_{\bar{\psi}} = [\mathbf{r}]_\psi/\eta_{\mathcal{F}}$ and $[r]_\psi \leq \mathbf{R}$.

Claim: Let $K = \{\, i \in I : [r_i]_{\theta_i} \in \mathrm{Sub}(\mathbf{R}_i) \,\}$. Then $K \in \mathcal{F}$.

Proof: First, if $\mathcal{B}$ is idempotent then $K = I$ which is automatically a member of $\mathcal{F}$. Now assume that the similarity type consists of finitely many operation symbols $f_1, \dots, f_m$. Then for any $i \in I$, the condition that $[r_i]_{\theta_i}$ be a subuniverse is equivalent to

$$\big(f_1(r, r, \dots, r)\ \theta_i\ r\big)\ \ \&\ \ \big(f_2(r, \dots, r)\ \theta_i\ r\big)\ \ \&\ \ \cdots\ \ \&\ \ \big(f_m(r, \dots, r)\ \theta_i\ r\big)$$

which in turn is equivalent to

$$i \in J(f_1(\mathbf{r}, \dots, \mathbf{r}), \mathbf{r}) \cap J(f_2(\mathbf{r}, \dots, \mathbf{r}), \mathbf{r}) \cap \cdots \cap J(f_m(\mathbf{r}, \dots, \mathbf{r}), \mathbf{r}).$$

But $[\mathbf{r}]_\psi$ is a subuniverse, so for each $j \leq m$, $J(f_j(\mathbf{r}, \dots, \mathbf{r}), \mathbf{r}) \in \mathcal{F}$. Hence $K = \bigcap_{j=1}^m J(f_j(\mathbf{r}, \dots, \mathbf{r}), \mathbf{r}) \in \mathcal{F}$.

Let $\mathcal{F}' = \{\, X \cap K : X \in \mathcal{F} \,\}$. Then one easily checks that $\mathcal{F}'$ is a filter on K. We shall show that

$$[\mathbf{r}]_\psi/\eta_{\mathcal{F}} \cong \prod_{k \in K} [r_k]_{\theta_k}/\eta_{\mathcal{F}'} \tag{4}$$

This will finish the proof since for $k \in K$, $[r_k]_{\theta_k} \in \mathrm{Sub}(\mathbf{R}_k)$, hence by assumption, $[r_k]_{\theta_k} \in \mathcal{A}$. Thus $[\bar{\mathbf{r}}]_{\bar{\psi}} = [\mathbf{r}]_\psi/\eta_{\mathcal{F}} \in \mathsf{P_r}(\mathcal{A}) \subseteq \mathcal{A}$.

Recall that if $\mathbf{x} \in [\mathbf{r}]_\psi$ then $J(\mathbf{x}, \mathbf{r}) \in \mathcal{F}$, hence $J(\mathbf{x}, \mathbf{r}) \cap K \in \mathcal{F}'$. For such an $\mathbf{x}$, define, for each $k \in K$

$$\tilde{x}_k = \begin{cases} x_k & \text{if } k \in J(\mathbf{x}, \mathbf{r}), \\ r_k & \text{otherwise.} \end{cases}$$

Notice that $\tilde{\mathbf{x}} \in \prod_K [r_k]_{\theta_k}$ and $\tilde{\mathbf{x}}$ agrees with $\mathbf{x}$ in "almost all" components.

Now define the map $g\colon [\mathbf{r}]_\psi \to \prod_K [r_k]_{\theta_k}/\eta_{\mathcal{F}'}$ by

$$g(\mathbf{x}) = \tilde{\mathbf{x}}/\eta_{\mathcal{F}'}.$$

g is easily seen to be a surjective homomorphism. We can finish the verification of (4) by showing that $\ker(g) = \eta_{\mathcal{F}}$ on $[\mathbf{r}]_\psi$. So let $\mathbf{x}, \mathbf{y} \in [\mathbf{r}]_\psi$. Then $\mathbf{x}\ \psi\ \mathbf{y}$ implies $J(\mathbf{x}, \mathbf{y}) \in \mathcal{F}$. Let $Z = \{\, k \in K : \tilde{x}_k = \tilde{y}_k \,\}$. Then

$$g(\mathbf{x}) = g(\mathbf{y}) \iff Z \in \mathcal{F}' \iff K \cap J(\mathbf{x}, \mathbf{y}) \cap Z \in \mathcal{F}.$$

But $K \cap J(\mathbf{x}, \mathbf{y}) \cap Z \subseteq [\![\mathbf{x} = \mathbf{y}]\!]$, so $[\![\mathbf{x} = \mathbf{y}]\!] \in \mathcal{F}$, hence $(\mathbf{x}, \mathbf{y}) \in \eta_{\mathcal{F}}$ as desired. $\square$

Theorem 10. *The Maltsev product of two quasivarieties of finite type is again a quasivariety. (If the second quasivariety is idempotent, the assumption of finite type can be dropped.)*

Proof. Combine Lemmas 8 and 9. □

Lemma 11. *If $\mathcal{A}$ and $\mathcal{B}$ are idempotent quasivarieties, then $\mathcal{A} \circ \mathcal{B}$ is idempotent.*

Proof. Let $\mathbf{R} \in \mathcal{A} \circ \mathcal{B}$ and $r \in R$. We must show that r is idempotent. Let θ be a pivot congruence on $\mathbf{R}$. Since $\mathcal{B}$ is idempotent, r/θ is an idempotent element of $\mathbf{R}/\theta \in \mathcal{B}$, so $[r]_\theta$ is a subuniverse of $\mathbf{R}$. Hence $[r]_\theta \in \mathcal{A}$. Since all members of $\mathcal{A}$ are idempotent and $r \in [r]_\theta$, r is an idempotent element. □

The noteworthy thing about idempotence is that every congruence class is a subuniverse. Thus when both $\mathcal{A}$ and $\mathcal{B}$ are idempotent, we can ignore the clause "$[r]_\theta \in \mathrm{Sub}(\mathbf{R})$" in the definition of Maltsev product.

Assume that $\mathcal{A}$ and $\mathcal{B}$ have finite similarity type, or that $\mathcal{B}$ is idempotent. Then $\mathcal{C} = \mathcal{A} \circ \mathcal{B}$ is a quasivariety, by Theorem 10. Let $\mathbf{F} = \mathbf{F}_{\mathcal{C}}(X)$ be a free $\mathcal{C}$-algebra over a set X. Then $\mathbf{F}/\lambda^F_{\mathcal{B}} \cong \mathbf{F}_{\mathcal{B}}(X)$, the free $\mathcal{B}$-algebra on X, [1, thm. 4.28]. Since $\lambda_{\mathcal{B}}$ can always serve as a pivot, we must have $[r]_\lambda \in \mathrm{Sub}(\mathbf{F}) \implies [r]_\lambda \in \mathcal{A}$. Unfortunately, there does not seem to be a natural way to view the algebra $[r]_\lambda$ as a homomorphic image of a free algebra on $\mathcal{A}$.

As a rule, the Maltsev product of two varieties need not be a variety (even in the idempotent case). However, if all congruences permute then we do indeed get a variety.

Theorem 12. *Let $\mathcal{A}$ and $\mathcal{B}$ be idempotent subvarieties of a quasivariety $\mathcal{C}$, and suppose that $\mathcal{C}$ is congruence-permutable* (*see* [1, pg. 122])*. Then $\mathcal{A} \circ_{\mathcal{C}} \mathcal{B}$ is a variety.*

Proof. By Theorem 10, we already know that the Maltsev product is closed under subalgebra and product, so the only thing left to show is closure under homomorphic images. For this let $\mathbf{R} \in \mathcal{A} \circ_{\mathcal{C}} \mathcal{B}$ and $\alpha \in \mathrm{Con}(\mathbf{R})$. We must show $\mathbf{R}/\alpha \in \mathcal{A} \circ_{\mathcal{C}} \mathcal{B}$. Let θ be an $\mathcal{A}$,$\mathcal{B}$-pivot on $\mathbf{R}$.

Let $\bar\theta = \theta \vee \alpha = \theta \circ \alpha$ (by congruence-permutability). We wish to show that $\bar\theta$ is an $\mathcal{A}$,$\mathcal{B}$-pivot on $\mathbf{R}/\alpha$, that is

$$(\mathbf{R}/\alpha)/(\bar\theta/\alpha) \in \mathcal{B} \text{ and} \tag{5}$$

$$r \in R \implies [r/\alpha]_{\bar\theta/\alpha} \in \mathcal{A}. \tag{6}$$

Note that we are tacitly appealing to idempotence in the formulation of (6). The first of these is easy. By the second isomorphism theorem [1, thm. 3.5], $(\mathbf{R}/\alpha)/(\bar\theta/\alpha) \cong \mathbf{R}/\bar\theta \in \mathsf{H}(\mathbf{R}/\theta) \subseteq \mathcal{B}$.

Now let $r \in R$ and set $\mathbf{A} = [r]_\theta$. $\mathbf{A}$ is a subalgebra of $\mathbf{R}$ by idempotence and $\mathbf{A} \in \mathcal{A}$ by assumption. Define

$$\mathbf{A}^\alpha = \bigcup_{a \in A} [a]_\alpha.$$

By the third isomorphism theorem [1, thm. 3.8]

$$\mathbf{A}^\alpha/(\alpha{\restriction_{A^\alpha}}) \cong \mathbf{A}/\alpha{\restriction_A} \in \mathcal{A}.$$

However, $\mathbf{A}^\alpha = [r]_{\bar\theta}$ since by congruence permutability

$$x \in A^\alpha \iff (\exists a \in R)\ x\ \alpha\ a\ \theta\ r \iff x\ \bar\theta\ r \iff x \in [r]_{\bar\theta}.$$

Finally, to verify (6) we need only observe that $[r/\alpha]_{\bar\theta/\alpha} = [r]_{\bar\theta}/\alpha = A^\alpha/\alpha$. □

Example 13 (Li, 2017)**.** Let $\mathcal{CIB}$ denote the variety of all commutative, idempotent binars, and let $\mathcal{Sq}$ be the variety of binars satisfying the identities

$$x^2 \approx x, \qquad xy \approx yx, \qquad x(xy) \approx y. \tag{7}$$

This is the variety of *squags*. Let $q(x,y,z) = y(xz)$. Then it is easy to check that q is a Maltsev term for $\mathcal{Sq}$ [1, thm. 4.64]. Now define the term

$$p(x,y,z) = \big(x(z(xy))\big)\cdot\big(z(x(zy))\big).$$

Then p is a Maltsev term for $\mathcal{Sq}\circ\mathcal{Sq}$.

Proof. let $\mathbf{A} \in \mathcal{Sq}\circ\mathcal{Sq}$. Thus, there is $\theta \in \mathrm{Con}(\mathbf{A})$ such that $\mathbf{A}/\theta \in \mathcal{Sq}$ and every $x/\theta \in \mathcal{Sq}$.

We shall show that $\mathbf{A} \models p(x,x,z) \approx z$, i.e., $(x(z(x^2)))(z(x(zx))) \approx z$. Let $w = x(zx)$. Since $\mathbf{A}/\theta \in \mathcal{Sq}$,

$$w/\theta = x/\theta \cdot \big(z/\theta \cdot x/\theta\big) = z/\theta$$

thus $w, z \in [z]_\theta \in \mathcal{Sq}$. But then (working in $[z]_\theta$) $p(x,x,z) \approx w(zw) \approx z$ as desired. The other identity, $p(x,z,z) \approx x$, is similar. □

Thus, by Theorem 12, $\mathcal{Sq}\circ\mathcal{Sq}$ is a variety. (Take $\mathcal{A} = \mathcal{B} = \mathcal{Sq}$ and $\mathcal{C} = \mathcal{Sq}\circ\mathcal{Sq}$.)

It would be interesting to find an equational base for $\mathcal{Sq}\circ\mathcal{Sq}$.

References

1. Clifford Bergman, *Universal algebra. Fundamentals and selected topics*, Pure and Applied Mathematics (Boca Raton), vol. 301, CRC Press, Boca Raton, FL, 2012. MR 2839398 (2012k:08001)
2. S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Springer-Verlag, New York, 1981, Available from `http://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html`.
3. J. Li, *Congruence n-permutable varieties*, Ph.D. thesis, Iowa State University, 2017, Graduate Theses and Dissertations. 15355.
4. Anatoliĭ Ivanovič Mal′cev, *Multiplication of classes of algebraic systems*, Siberian Math. J. **8** (1967), 254–267, Translated in [5].
5. ———, *The metamathematics of algebraic systems. Collected papers: 1936–1967*, North-Holland Publishing Co., Amsterdam-London, 1971, Translated, edited, and provided with supplementary notes by Benjamin Franklin Wells, III, Studies in Logic and the Foundations of Mathematics, Vol. 66. MR 0349383

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